

# A relation between diffusion, temperature and the cosmological constant

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## Abstract

We show that the temperature of a diffusing fluid with the diffusion constant  $\kappa^2$  in an expanding universe approaches a constant limit  $T_\infty = \frac{\kappa^2}{H}$  in its final de Sitter stage characterized by the horizon  $\frac{1}{H}$  determined by the Hubble constant. If de Sitter surface temperature in the final equilibrium state coincides with the fluid temperature then the cosmological constant  $\Lambda = 3H^2 = 6\pi\kappa^2$ .

## 1 Introduction

Since the accelerated expansion has been discovered [1][2] the  $\Lambda$ CDM model became the standard model of cosmology. There are many aspects of this model which still need exploration, e.g., the relation between the dark matter (DM) and dark energy (DE) densities (coincidence). Such properties can be explained in models with some interactions between universe constituents. In [3] we have suggested a model in which the DM energy-momentum non-conservation is a consequence of its diffusive energy gain from an environment described by an ideal fluid (as a model of dark energy). In such a scheme equations of the relativistic diffusion have as a solution a phase space distribution in the form of the Jüttner distribution [4] with a time-dependent temperature. The scale factor can increase exponentially for a large time in Einstein gravity of such a diffusive fluid [5] in agreement with the  $\Lambda$ CDM prediction. In a complete theory baryons and radiation will be described as excitations of quantum fields which are to live in a space-time approaching at large time de Sitter space-time. It is known that if quantum fields at finite temperature  $T$  are to exist in de Sitter space then there must be a relation between the temperature and the de Sitter

horizon [6][7]. A simple reason is that at an imaginary time the model of de Sitter space as a pseudo-sphere becomes the sphere. The finite temperature quantum fields to be periodic in the imaginary time (KMS condition) must be periodic on the sphere. Hence,  $\beta = \frac{1}{T} = 2\pi r$  where  $r = \frac{1}{H}$  is the radius of the sphere. This relation can be considered as an extension of the black hole horizon-temperature correspondence [8][9]. In de Sitter case the relation has been confirmed by Gibbons and Hawking [10] through a calculation of the temperature of particles created from the event horizon. In this paper we assume that DM consists of relativistic diffusing particles ( see [3] [11][12]). The relativistic diffusion can be defined in a coordinate independent way as a diffusion on the tangent bundle [13]. We restrict ourselves to the study of the relativistic diffusion in a homogenous space-time. We show that solutions of the diffusion equation approach an equilibrium limit which depends on the diffusion constant and the Hubble constant. Assuming that DM and baryonic matter (described by quantum fields) coexist in de Sitter space we obtain the relation  $H^2 = 2\pi\kappa^2$  between the diffusion constant  $\kappa^2$  and the Hubble constant  $H$ . Finally, we point out that if at large time the diffusion becomes non-relativistic then our conclusion concerning the relation between the diffusion constant and the cosmological constant does not change.

## 2 Diffusive DM-DE interaction

We consider an energy-momentum tensor  $T^{\mu\nu}$  consisting of a separately conserved baryonic part and  $T_D^{\mu\nu}$  describing the dark sector. In order to approach the coincidence problem we assume an interaction between DM and DE. If  $T_D^{\mu\nu} = T_M^{\mu\nu} + T_E^{\mu\nu}$  is to be conserved then we have a relation between the non-conservation laws of DE and DM

$$-\nabla_\mu T_E^{\mu\nu} = \nabla_\mu T_M^{\mu\nu} \equiv 3\kappa^2 J^\nu, \quad (1)$$

where  $T_M^{\mu\nu}$  is the energy-momentum of DM and  $T_E^{\mu\nu}$  corresponds to DE. The rhs of eq.(1) could be treated as a definition of  $J^\nu$ ,  $\kappa^2$  is a parameter which measures the strength of the DM-DE interaction. Our crucial assumption is that the four-vector  $J^\nu$  appearing on the rhs of eq.(1) is conserved

$$\nabla_\mu J^\mu = 0. \quad (2)$$

Such a conserved current can describe a flow of particles with the density  $J^0$ . If additionally we assume that the particle current is realized as a stream of particles with the phase space distribution  $\Omega(p, x)$  then the current should be expressed as

$$J^\mu = \sqrt{g} \int \frac{d\mathbf{p}}{p^0} p^\mu \Omega. \quad (3)$$

The energy-momentum of a stream of particles is determined by the formula

$$T_M^{\mu\nu} = \sqrt{g} \int \frac{d\mathbf{p}}{p^0} p^\mu p^\nu \Omega. \quad (4)$$

In eqs.(3)-(4)  $g_{\mu\nu}$  is the Riemannian metric, the momenta satisfy the mass-shell condition  $g_{\mu\nu} p^\mu p^\nu = m^2$  (the velocity of light  $c = 1$ ) and  $g = |\det g_{\mu\nu}|$ . The definitions (3)-(4) come from relativistic dynamics.  $J^\nu$  and  $T_M^{\mu\nu}$  are conserved as a consequence of the Liouville equation if the number of particles is preserved and there is no exchange of energy with the environment, i.e.,  $\kappa^2 = 0$ . We wish to find a realization of the conservation laws (1)-(2). If the phase space distribution  $\Omega$  is to satisfy a differential equation which is to reduce to the Liouville equation when  $\kappa^2 = 0$  then the modified differential equation must be a diffusion equation. The diffusion is determined in the unique way by the requirement that the diffusing particle moves on the mass-shell (see [14][13][15][16]).

Let  $p^j$  be spatial coordinates on the mass-shell  $p^2 = m^2$ . We define the Riemannian metric on the mass-shell as the one induced from the pseudo-Riemannian metric  $g_{\mu\nu}$  on the space-time

$$ds^2 = g_{\mu\nu} dp^\mu dp^\nu = -m^2 G_{jk} dp^j dp^k,$$

where  $p^0$  is expressed by  $p^j$ .

The inverse matrix is (we assumed that  $g_{0k} = 0$ )

$$G^{jk} = -g^{jk} m^2 + p^j p^k. \quad (5)$$

Next,

$$G \equiv \det(G_{jk}) = m^{-4} \det(g_{jk}) \omega^{-2},$$

where

$$\omega^2 = m^2 - g_{jk} p^j p^k.$$

We define diffusion as a stochastic process generated by the Laplace-Beltrami operator  $\Delta_H^m$  on the mass-shell

$$\Delta_H^m = \frac{1}{\sqrt{G}} \partial_j G^{jk} \sqrt{G} \partial_k, \quad (6)$$

where  $\partial_j = \frac{\partial}{\partial p^j}$ .

The transport equation for the diffusion generated by  $\Delta_H$  reads

$$(p^\mu \partial_\mu^x - \Gamma_{\mu\nu}^k p^\mu p^\nu \partial_k) \Omega = \kappa^2 \Delta_H^m \Omega, \quad (7)$$

where  $\kappa^2$  is the diffusion constant and  $\partial_\mu^x = \frac{\partial}{\partial x^\mu}$ . Both operators: the infinitesimal transport on the lhs of eq.(7) as well as the generator of diffusion on the rhs preserve the mass-shell. The operator  $-\Delta_H^m$  is self-adjoint and non-negative in the Hilbert space  $L^2(\frac{d\mathbf{p}}{p^0})$ . Eq.(7) does not depend on the choice of coordinates

on the tangent bundle [13]. It can be checked by direct calculations using eq.(7) that  $\nabla_\mu T_M^{\mu\nu} = 3\kappa^2 J^\nu$ . For the first part of the equality (1) we need a model of the dark energy. In [3] we have chosen an ideal fluid as a model of  $T_E^{\mu\nu}$ . We have determined its energy density  $\rho_{de}$  from eq.(1). The pressure  $p_{de}$  is defined as  $p_{de} = w\rho_{de}$  where  $w$  is a free parameter. The Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{3}(T_M^{\mu\nu} + T_E^{\mu\nu})$$

(where  $R_{\mu\nu}$  is the Ricci tensor) have asymptotic exponentially growing solutions if  $w = -1$ .

### 3 Solutions of the diffusion equation

We restrict our discussion to the metric

$$ds^2 = dt^2 - a^2 d\mathbf{x}^2. \quad (8)$$

We change variables

$$\mathbf{p} = a^{-1}\mathbf{q}.$$

Let

$$\omega_1 = \sqrt{m^2 + \mathbf{q}^2}. \quad (9)$$

Then, the diffusion equation (7)(neglecting the spatial dependence) reads

$$(\partial_t - Hq^k \partial_k)\Omega = \kappa^2 \partial_k \left( \omega_1^{-1} (m^2 \delta^{jk} + q^j q^k) \partial_j \right) \Omega, \quad (10)$$

where  $H = a^{-1} \partial_t a$  and  $\partial_j = \frac{\partial}{\partial q^j}$ .

We cannot solve eq.(10) explicitly in general. However, if  $H = \text{const}$ , i.e.  $a = \exp(Ht)$ , then we obtain a particular solution (we have got this solution in [17] using slightly different notation)

$$\Omega^{deS}(q) = \exp(-3Ht) \exp\left(-\frac{H\omega_1(q)}{\kappa^2}\right). \quad (11)$$

The time-dependent factor in eq.(11) cancels if we calculate expectation values of observables  $\mathcal{O}$  with respect to the normalized distribution

$$\langle \mathcal{O} \rangle_\Omega = \left( \int d\mathbf{q} \Omega \right)^{-1} \int \frac{d\mathbf{q}}{\omega_1} \Omega \omega_1 \mathcal{O} \equiv (\Omega, \omega_1)^{-1} (\Omega, \omega_1 \mathcal{O}), \quad (12)$$

where on rhs of eq.(12) we introduce a scalar product in the Hilbert space  $L^2(\frac{d\mathbf{q}}{\omega_1})$  of functions square integrable with respect to the Lorentz invariant measure on the mass shell (in this Hilbert space the diffusion generator is self-adjoint).

The solution (11) describes the Jüttner equilibrium distribution [4] corresponding to the temperature (we use the units with the Boltzmann constant  $k_B = 1$  and  $\hbar = 1$ )

$$T_\infty = \frac{\kappa^2}{H}. \quad (13)$$

The Jüttner distribution is the canonical distribution determined by the maximum entropy principle (assuming the fixed value of the energy  $\langle p^0 \rangle$ ). The entropy is defined by the integral (in terms of the original momenta  $\mathbf{p}$  of sec.2)

$$S = - \int d\mathbf{x} g d\mathbf{p} \Omega \ln \Omega \quad (14)$$

In [5] we have considered the time evolution of the Jüttner distribution (we choose  $a(t_0) = 1$  and we denote  $T_0 = T(t_0)$ )

$$\Omega(t = t_0, T_0, q) = \exp\left(-\frac{|\mathbf{q}|}{T_0}\right) \quad (15)$$

for an arbitrary scale factor  $a$  in the limit  $m = 0$ . We have shown that the solution of eq.(7) with the initial condition (15) is again the Jüttner distribution

$$\Omega^{(m=0)}(t) = T_0^3 (T_0 + \kappa^2 A)^{-3} \exp\left(-\frac{a}{T_0 + \kappa^2 A} |\mathbf{q}|\right), \quad (16)$$

where

$$A(t) = \int_{t_0}^t a(s) ds. \quad (17)$$

The temperature corresponding to the phase space distribution  $\Omega(t)$  can again be determined either from the maximum entropy principle as the factor in front of  $|\mathbf{q}|$  in eq.(16) or by calculating the entropy (14) and the energy  $E$  defined by

$$E = \int d\mathbf{x} \sqrt{g} T_M^{00} \quad (18)$$

Then, by general rules of statistical physics

$$\frac{1}{T} = \frac{\partial S}{\partial E} \quad (19)$$

The result is

$$T(t) = \frac{T_0 + \kappa^2 A}{a}. \quad (20)$$

If for a large time  $a(t) \simeq \exp(\lambda t)$  then  $\Omega^{(m=0)} \rightarrow T_0^3 (\frac{\lambda}{\kappa^2})^3 \Omega^{deS}$  (with  $m = 0$  and  $\lambda = H$  in eq.(11)). Moreover, if both  $A(t)$  and  $a(t)$  tend to infinity at large  $t$  and the limit of  $T(t)$  in eq.(20) is  $\frac{\lambda}{\kappa^2}$ , then,  $\lim_{t \rightarrow \infty} a(t) \exp(-\lambda t) = \text{const} \neq 0$ . Hence, a finite limit of the expectation value (12) with  $\Omega^{(m=0)}$  implies the de Sitter growth of  $a$ .

If instead of the Jüttner distribution (15) we take as an initial condition a superposition of Jüttner distributions ( $\mu \geq 0$ )

$$\Omega(t_0, q) = \int_0^\infty dT_0 \mu(T_0) \exp(-\frac{|\mathbf{q}|}{T_0}) \quad (21)$$

then

$$\Omega(t, q) = \int_0^\infty dT_0 \mu(T_0) T_0^3 (T_0 + \kappa^2 A)^{-3} \exp\left(-\frac{a}{T_0 + \kappa^2 A} |\mathbf{q}| \right) \quad (22)$$

If we allow complex  $T_0$  then eq.(21) represents a large class of initial conditions (including Fourier transforms) and eq.(22) a general solution of the diffusion equation with  $m = 0$ .

Assuming an exponential growth of  $a$  (as discussed below eq.(20)) and using Lebesgue dominated convergence theorem we obtain a limiting behaviour of the solution (22)

$$\begin{aligned} \lim_{t \rightarrow \infty} \exp(3Ht) \Omega(t, q) \\ = \exp(-\frac{H|\mathbf{q}|}{\kappa^2}) \int_0^\infty dT_0 \mu(T_0) T_0^3 (\frac{H}{\kappa^2})^3 \end{aligned} \quad (23)$$

if  $\int_0^\infty dT_0 \mu(T_0) T_0^3 < \infty$ . The exponential factor  $\exp(3Ht)$  does not contribute to the expectation value (12). Hence, in the sense of the expectation values the limit  $t \rightarrow \infty$  of the solution of the relativistic diffusion equation with the initial condition (21) is again the Jüttner distribution with the temperature  $\frac{\kappa^2}{H}$ .

We are unable to derive an explicit solution of the diffusion equation for arbitrary  $a$  and  $m > 0$ . We apply a perturbative method. We consider the initial condition

$$\Omega(t_0, q) = \int_0^\infty dT_0 \mu(T_0) \exp(-\frac{\omega_1(q)}{T_0}). \quad (24)$$

We look for solutions of eq.(7) with  $m \neq 0$  in the form

$$\Omega(t, q) = \int_0^\infty dT_0 \mu(T_0) L(t) \exp(-\omega_1 \alpha(t)) f_t(|\mathbf{q}|, T_0) \quad (25)$$

with

$$\alpha(t) = \frac{a}{T_0 + \kappa^2 A} \quad (26)$$

and

$$L(t) = T_0^3 (T_0 + \kappa^2 A)^{-3}. \quad (27)$$

We would like to show that similarly as in the massless case (22) if the space-time asymptotically tends to de Sitter space then  $\Omega_t \rightarrow \Omega^{deS}$  (11).

Inserting eq.(25) in eq.(7) we obtain an equation for  $f$

$$\partial_t f - \kappa^2 \omega_1 \partial_q^2 f - \kappa^2 (\kappa^{-2} H q - 2\alpha q + \frac{3q}{\omega_1} + \frac{2m^2}{q\omega_1}) \partial_q f = \frac{m^2}{\omega_1} \partial_t \alpha f, \quad (28)$$

where  $\partial_q = \frac{\partial}{\partial q}$  and  $q = |\mathbf{q}|$ . Let us note that if  $a(t) = \exp(\lambda t) + r(t)$ , where  $r(t)$  is a slowly varying function, then

$$H = (\exp(\lambda t) + r(t))^{-1}(\lambda \exp(\lambda t) + \partial_t r(t)) \simeq \lambda + \exp(-\lambda t)R(t),$$

where  $R(t)$  is a slowly varying function. Hence,  $H \simeq \lambda$  and we can treat  $H$  as a constant up to an exponentially small correction. For the same reason  $\alpha - \kappa^{-2}H = \exp(-Ht)R_1$  where  $R_1$  is a slowly varying function and  $\partial_t \alpha = \exp(-Ht)R_2$  where  $R_2$  is a slowly varying function. The exponentially decaying corrections will be shifted to the rhs of eq.(28). Now, eq.(28) can be expressed as

$$\partial_t f + \mathcal{M}f = (v(t, q)\partial_q + \frac{m^2}{\omega_1}\partial_t \alpha)f, \quad (29)$$

where

$$-\mathcal{M} = \kappa^2 \omega_1 \partial_q^2 + \kappa^2 (-\kappa^{-2}Hq + \frac{3q}{\omega_1} + \frac{2m^2}{q\omega_1})\partial_q. \quad (30)$$

$\mathcal{M}$  depends neither on time nor on  $T_0$ . The functions  $v(t, q)$  and  $\partial_t \alpha$  on the rhs of eq.(29) are decreasing as  $\exp(-Ht)$  times a slowly varying function. We can rewrite eq.(29) as an integral equation with the initial value  $f_0 = 1$  (i.e., we assume the initial condition (24)).

$$f_t(q) = 1 + \int_{t_0}^t \exp(-(t-s)\mathcal{M}) \left( \exp(-Hs)R_3(s)q\partial_q + \frac{m^2}{\omega_1} \exp(-Hs)R_4(s) \right) f_s ds, \quad (31)$$

where  $R_k$  are slowly varying functions. We solve eq.(31) by iteration. We can show that in the sense of the convergence of expectation values (12) owing to the exponential decrease of the integrand on the rhs of eq.(31) each term of the iteration series tends to zero as  $t \rightarrow \infty$ . Hence,  $f_t \rightarrow 1$ .

Let us still consider the non-relativistic limit of the diffusion equation (7). In the limit

$$m^2 a^2 \rightarrow \infty$$

eq.(7) reads [5][3]

$$m^{-1}\kappa^{-2}(\partial_t - Hq^j\partial_j)\Omega = \Delta_{\mathbf{q}}\Omega. \quad (32)$$

$\Delta_{\mathbf{q}}$  is the Laplacian and

$$A_{NR} = 2 \int_{t_0}^t ds a^2. \quad (33)$$

A solution of the diffusion equation (32) which starts from the Maxwell-Boltzmann distribution (the non-relativistic approximation of the Jüttner distribution) is [5][3]

$$\Omega_{NR}(t, \mathbf{q}) = T_0^{\frac{3}{2}} (\kappa^2 A_{NR} + T_0)^{-\frac{3}{2}} \exp \left( -a^2 \frac{\mathbf{q}^2}{2m(\kappa^2 A_{NR} + T_0)} \right). \quad (34)$$

Using eq.(34) we can solve the diffusion equation with an initial condition

$$\Omega(t_0, \mathbf{q}) = \int_0^\infty dT_0 \mu(T_0) \exp\left(-\frac{\mathbf{q}^2}{2mT_0}\right). \quad (35)$$

The solution reads

$$\Omega_{NR}(t, \mathbf{q}) = \int_0^\infty dT_0 \mu(T_0) \Omega_{NR}(t, \mathbf{q}). \quad (36)$$

It follows from eq.(34) and from eq.(19) that in the non-relativistic limit the temperature is

$$T(t) = \frac{T(t_0) + \kappa^2 A_{NR}}{a^2}. \quad (37)$$

If for a large time  $a(t) \simeq \exp(Ht)$  (plus a slowly varying function) then

$$\begin{aligned} & \exp(3Ht) \Omega_{NR}(t, \mathbf{q}) \\ & \rightarrow \exp\left(-\frac{H\mathbf{q}^2}{2m\kappa^2}\right) \left(\frac{H}{\kappa^2}\right)^{\frac{3}{2}} \int_0^\infty dT_0 \mu(T_0) T_0^{\frac{3}{2}} \end{aligned} \quad (38)$$

Hence, the limiting temperature is the same as in the relativistic case (13)  $T(t) \rightarrow T_\infty = \kappa^2 H^{-1}$  (we obtain the result (38) even if the temperature  $T_0$  in eqs.(34)-(35) is a complex number ).

If we assume that the diffusing dark matter is in equilibrium with the baryonic matter described by quantum fields defined on the de Sitter space [6][10] then  $T_\infty = T_{deS} = \frac{H}{2\pi}$ . Hence, from eq.(13)

$$H^2 = 2\pi\kappa^2 \quad (39)$$

## 4 Conclusions

We have shown for a class of initial conditions that (in the sense of expectation values of observables) solutions of the diffusion equation in an exponentially expanding universe tend to the Jüttner equilibrium distribution with the temperature  $T_\infty = \frac{\kappa^2}{H}$ . It is widely recognized that quantum field theory at finite temperature on de Sitter space has the temperature related to the event horizon. In a generalized thermodynamics [8][18][19][20][21] the second law leads to the conclusion that the temperature inside the horizon must be bigger than the horizon temperature which is achieved when the system is approaching the equilibrium. In a classical approximation the energy momentum of the quantum (baryonic) fields at the temperature  $T_\infty$  will be described by classical fluids of temperature  $T_\infty$ . In our model the DM energy-momentum non-conservation results from a diffusion in a fluid of DE. If we accept this scheme then the diffusion constant in an expanding universe is determined by the Hubble constant (or vice versa). Inserting the current value of the Hubble constant we obtain



$\kappa \simeq 10^{-42} GeV$ . The small value of the diffusion constant follows from the weak interaction between DM and DE fluids. The weakness of this interaction could be regarded as an answer,  $\Lambda = 6\pi\kappa^2$ , to the cosmological constant problem [22].

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